

Explicit Superconic Curves

Sunggoo Cho*

School of Computer Science, Semyung University, Chechon, Chungbuk 390 - 711, Korea

(Dated: January 14, 2016)

Conics and Cartesian ovals are very important curves in various fields of science. Also aspheric curves based on conics are useful in optics. Superconic curves recently suggested by A. Greynolds are extensions of both conics and Cartesian ovals and have been applied to optics while they are not extensions of aspheric curves based on conics.

In this work, we investigate another kind of superconic curves that are extensions of not only conics and Cartesian ovals but also aspheric curves based on conics. Moreover, the superconic curves are represented in explicit form while Greynolds' superconic curves are in implicit form.

PACS numbers: 02.30.Gp, 02.60.Cb, 42.15.Dp

I. INTRODUCTION

Conics and Cartesian ovals are very useful curves in science[1–4]. Especially, in optical design, the conic curves with curvature c_0 and conic constant K are described in implicit and explicit form as follows[5, 6]:

$$(1 + K)c_0z^2 - 2z + c_0y^2 = 0, \quad (1)$$

and

$$z = c_0y^2 / \{1 + [1 - (1 + K)c_0^2y^2]^{\frac{1}{2}}\}, \quad (2)$$

usually in (z, y) -plane for representing rotationally symmetric surfaces about the z -axis. We note that the explicit form like Eq.(2) is essential in optical design since it can be interpolated even in the region where the curve is not defined. For an example, there is no solution z of a circle of radius 1 when $y > 1$, which is the case when the square root term $1 - (1 + K)c_0^2y^2$ in the denominator becomes negative. In this case, we may interpolate the circle as parabola $z = c_0y^2$ by putting the square root term to be zero for $y > 1$. Such interpolations enable us to construct so called *an aspheric curve based on conic* that is usually represented for any y in explicit form [6] as

$$z = c_0y^2 / \{1 + [1 - (1 + K)c_0^2y^2]^{\frac{1}{2}}\} + \sum_{n=2}^N f_{2n}y^{2n} \quad (3)$$

for some positive integer N and coefficients f_{2n} 's.

Another useful curves are Cartesian ovals[2, 3]. A Cartesian oval is defined as the set of points such that the sum of whose weighted distances from two fixed points is a constant. In general, Cartesian ovals are quartic equations and they are famous for their perfectly focusing refraction property in optics[7–10].

On the other hand, superconic curves[11] were suggested by A. Greynolds as extensions of conics and Cartesian ovals, which are expressed in implicit form as

$$c_0Kz^2 - 2(1 + b_1s^2 + b_2s^4 + \cdots)z + (c_0 + c_1s^2 + c_2s^4 + \cdots)s^2 = 0, \quad (4)$$

for some constants b_1, b_2, \cdots and c_1, c_2, \cdots . Here the parameter s is defined by $s^2 \equiv z^2 + y^2$. Hence it is obvious that superconic curves are conics if $b_i = 0$ and $c_i = 0$ for $i = 1, 2, \cdots$. If $b_1 \neq 0$ and $c_1 \neq 0$ while $b_i = 0$ and $c_i = 0$ for $i = 2, \cdots$, they are Cartesian ovals. However, it is obvious that the superconic curves cannot be extensions of aspheric curves based on conics of Eq.(3). Furthermore, it seems to be hard to express Eq.(4) in explicit form since $s^2 = z^2 + y^2$. In fact, Greynolds seemed to give up a closed-form explicit representation corresponding to the implicit form as he stated in his work[11].

In this work, we shall investigate another kind of superconic curves, which are extensions of both conic and Cartesian oval curves like Greynolds'. On the contrary to Greynolds', however, our superconic curves are not only extensions of aspheric surfaces based on conics of Eq.(3) but also expressed in explicit form.

*Electronic address: sgcho@semyung.ac.kr, oldrock9@hanmail.net

In this work we are interested in a solution that passes through the origin in (z, y) -plane among four solutions of the quartic equation described by a Cartesian oval.

Definition I.1 *An optical solution is defined to be the solution that passes through the origin among solutions of a Cartesian oval.*

The main strategy of this work is to find the optical solution such that it is not only expressed in an appropriate explicit form whose limit is conic but also interpolated in the region where the Cartesian oval is not defined.

Of course, any quartic equations can be solved explicitly in general by the method of L. Ferrari who is attributed with the discovery of the explicit solutions to the quartic equations in 1540, and they are still studied for their diverse solving methods and properties[12–14]. However, the usually known general explicit forms for the solutions of quartic equations do not seem to be appropriate for our purpose.

In section II, we shall decompose a Cartesian oval into a product of two specific quadratic forms, two solutions of which are the candidates for the optical solution with an appropriate form for the limit and interpolation. In section III, we shall investigate the initial criteria for the choice of the optical solution between the candidates. Moreover, the continuity and interpolation of the optical solution shall be discussed. In section IV, we shall show that conics are the limiting cases of optical solutions from a different point of view than are usually known in the literature[3, 15, 16]. The limiting process in this section gives us more insights on the relationship between optical solutions and conics. The main result of this work about superconic curves comes from the limiting relationship. Finally, in section V, we shall discuss about a family of curves including optical solutions and conics and demonstrate an example. A lot of computational work are required in this paper and done using Mathematica.

II. THE CANDIDATES FOR THE OPTICAL SOLUTIONS FROM CARTESIAN OVALS

A. Motivations and notations for Cartesian ovals

There are several forms for Cartesian ovals in the literature[2, 3, 7]. The purpose of this section is to find an appropriate form for the quartic equation of a Cartesian oval such that not only it yields the optical solution but also the optical solution can be interpolated in the region where the Cartesian oval is not defined. Moreover, the optical solution should become a conic curve as its special limiting case.

For this purpose, let us describe a Cartesian oval from a physical point of view that gives us the natural motivation for the Cartesian oval. Among others, it may be better to start from the Snell's law of refraction on a curve that passes through the origin in the (z, y) -plane:

$$n \sin(\phi - \theta) = n' \sin(\phi - \theta'), \quad (5)$$

where n and n' are the refractive indices, θ and θ' are the angles which the incident and refracted ray make with the $+z$ axis respectively, and ϕ is defined by $\tan \phi \equiv -dz/dy$. Now let (z_i, y_i) be the position of a point light source and (z_o, y_o) be a focusing point. Also define signs s_i, s_o as follows: if $z_i < z$, $s_i = -1$, otherwise $s_i = +1$. Similarly, if $z_o < z$, $s_o = -1$, otherwise $s_o = +1$.

Then the law of refraction on a perfectly focusing curve[15] may be written as

$$\begin{aligned} \frac{dz}{dy} &= (-1) \frac{n' \sin \theta' - n \sin \theta}{n' \cos \theta' - n \cos \theta} \\ &= (-1) \frac{s_o n' (y_o - y) \sqrt{(z_i - z)^2 + (y_i - y)^2} - s_i n (y_i - y) \sqrt{(z_o - z)^2 + (y_o - y)^2}}{s_o n' (z_o - z) \sqrt{(z_i - z)^2 + (y_i - y)^2} - s_i n (z_i - z) \sqrt{(z_o - z)^2 + (y_o - y)^2}}, \end{aligned} \quad (6)$$

which is the first order exact differential equation whose solution is given by

$$s_i n \sqrt{(z - z_i)^2 + (y - y_i)^2} - s_o n' \sqrt{(z - z_o)^2 + (y - y_o)^2} = n \kappa, \quad (7)$$

where κ is a constant to be determined by the initial position of the curve.

Eq.(7) is one of many different forms of Cartesian ovals. Without loss of generality, we assume that $y_i = y_o = 0$ and define $m \equiv n'/n > 0$. Thus a Cartesian oval is described as the set of points (z, y) satisfying the following equation:

$$s_i [(z - z_i)^2 + y^2]^{\frac{1}{2}} - m s_o [(z - z_o)^2 + y^2]^{\frac{1}{2}} = \kappa. \quad (8)$$

We also assume that it passes through the origin $(0, 0)$. Then $\kappa = z_i - m z_o$.

Now let us define $\eta_i \equiv \epsilon/z_i$ and $\eta_o \equiv \epsilon/z_o$ for a constant $\epsilon > 0$ since the notations are convenient for later use. Then Eq.(8) may be written as

$$\eta_o[(\eta_i z - \epsilon)^2 + (\eta_i y)^2]^{\frac{1}{2}} - m\eta_i[(\eta_o z - \epsilon)^2 + (\eta_o y)^2]^{\frac{1}{2}} = \eta_i \eta_o \kappa \equiv k \quad (9)$$

with $k = \epsilon(\eta_o - m\eta_i)$.

In the next subsection, we shall decompose a Cartesian oval described by Eq.(9) into the product of two quadratic factors, two solutions of which are the candidates for the optical solution.

B. The candidates for the optical solution

Now we are interested in a quartic equation described by Eq.(9). Thus in order to avoid those cases for which Eq.(9) becomes quadratic, we assume that $k \neq 0$, $m \neq 1$, $\eta_i \neq \eta_o$, $\eta_i \neq 0$ and $\eta_o \neq 0$.

Now we define $x \equiv \eta_i \eta_o z$. Then Eq.(9) may be rewritten as follows.

$$s_o[(x - \eta_o \epsilon)^2 + (\eta_i \eta_o y)^2]^{\frac{1}{2}} = k + m s_i[(x - \eta_i \epsilon)^2 + (\eta_i \eta_o y)^2]^{\frac{1}{2}}. \quad (10)$$

Now it is easy to see that Eq.(10) can be transformed into the following specific form by squaring twice to remove the two root terms

$$(b_2 x^2 + b_1 x + b_0)^2 = a_2 x^2 + a_1 x + a_0. \quad (11)$$

where

$$\begin{aligned} a_2 &\equiv 4k^2 m^2, \\ a_1 &\equiv -2a_2 \epsilon \eta_i, \\ a_0 &\equiv a_2 \epsilon^2 \eta_i^2 + a_2 \eta_i^2 \eta_o^2 y^2, \end{aligned} \quad (12)$$

and

$$\begin{aligned} b_2 &\equiv 1 - m^2, \\ b_1 &\equiv -2\epsilon(\eta_o - m^2 \eta_i), \\ b_0 &\equiv \epsilon^2(\eta_o^2 - m^2 \eta_i^2) - k^2 + (1 - m^2)\eta_i^2 \eta_o^2 y^2. \end{aligned} \quad (13)$$

Then we add both sides with $(b_2 x^2 + b_1 x + b_0 + \lambda)^2 - (b_2 x^2 + b_1 x + b_0)^2$ to obtain

$$(b_2 x^2 + b_1 x + b_0 + \lambda)^2 = (a_2 + 2\lambda b_2)(x + \frac{a_1 + 2\lambda b_1}{2(a_2 + 2\lambda b_2)})^2, \quad (14)$$

where λ is supposed to satisfy, unless $a_2 + 2\lambda b_2 = 0$,

$$(a_0 + 2\lambda b_0) + \lambda^2 = \frac{(a_1 + 2\lambda b_1)^2}{4(a_2 + 2\lambda b_2)}. \quad (15)$$

Thus we have a resolvent cubic equation[12] in λ

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \quad (16)$$

where the coefficients, after divided by 8, are given by

$$\begin{aligned} a &\equiv b_2, \\ b &\equiv (a_2 + 4b_2 b_0 - b_1^2)/2, \\ c &\equiv a_2 b_0 + a_0 b_2 - a_1 b_1/2, \\ d &\equiv a_2 a_0/2 - a_1^2/8. \end{aligned} \quad (17)$$

In terms of m, η_i and η_o , Eq.(17) is written as

$$\begin{aligned} a &= 1 - m^2, \\ b &= 2\epsilon^2(m - 1)[\eta_o^2(1 + m) + 2\eta_i^2 m^2(1 + m) - 2\eta_i \eta_o m(1 + 2m)] + 2\eta_i^2 \eta_o^2 (m^2 - 1)^2 y^2, \\ c &= 4\epsilon^4 m^2(1 - m)\eta_i(\eta_i - 2\eta_o + m\eta_i)(\eta_o - m\eta_i)^2 + 8\epsilon^2 m^2(1 - m^2)\eta_i^2 \eta_o^2 (\eta_o - m\eta_i)^2 y^2, \\ d &= 8\epsilon^4 m^4 \eta_i^2 \eta_o^2 (\eta_o - m\eta_i)^4 y^2. \end{aligned} \quad (18)$$

Using the root λ of Eq.(16), Eq.(14) becomes

$$\begin{aligned}
0 &= \{b_2x^2 + [b_1 + (a_2 + 2\lambda b_2)^{\frac{1}{2}}]x + b_0 + \lambda + \frac{a_1 + 2\lambda b_1}{2(a_2 + 2\lambda b_2)^{\frac{1}{2}}}\} \\
&\quad \times \{b_2x^2 + [b_1 - (a_2 + 2\lambda b_2)^{\frac{1}{2}}]x + b_0 + \lambda - \frac{a_1 + 2\lambda b_1}{2(a_2 + 2\lambda b_2)^{\frac{1}{2}}}\} \\
&\equiv (b_2x^2 + p_+x + q_+)(b_2x^2 + p_-x + q_-),
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
p_{\pm} &\equiv b_1 \pm (a_2 + 2\lambda b_2)^{\frac{1}{2}}, \\
q_{\pm} &\equiv b_0 + \lambda \pm \frac{a_1 + 2\lambda b_1}{2(a_2 + 2\lambda b_2)^{\frac{1}{2}}}.
\end{aligned} \tag{20}$$

If we suppose that $p_1, p_2 \neq 0$ and replace x by z in Eq.(19), we have the generic form for the quartic equation

$$(A_+z^2 - 2z + B_+)(A_-z^2 - 2z + B_-) = 0, \tag{21}$$

where

$$A_{\pm} \equiv \frac{-2b_2\eta_i\eta_o}{p_{\pm}} \text{ and } B_{\pm} \equiv \frac{-2q_{\pm}}{\eta_i\eta_o p_{\pm}}. \tag{22}$$

Here the two solutions $z = B_{\pm}/[1 - (1 - A_{\pm}B_{\pm})^{\frac{1}{2}}] = [1 + (1 - A_{\pm}B_{\pm})^{\frac{1}{2}}]/A_{\pm}$ are excluded since they do not pass through the origin, which may be easily seen from the fact that A_{\pm} are finite under the condition that $p_{\pm} \neq 0$.

Thus, we obtain the two candidates for the optical solution from the quartic equation in Eq.(21).

Lemma II.1 *Let $p_{\pm} \neq 0$, $a_2 + 2\lambda b_2 \neq 0$ and $(1 - A_{\pm}B_{\pm})^{\frac{1}{2}} \geq 0$. Then the optical solution of a Cartesian oval in Eq.(9) is given by one of the followings:*

$$z = \frac{B_+}{1 + (1 - A_+B_+)^{\frac{1}{2}}}, \tag{23a}$$

$$z = \frac{B_-}{1 + (1 - A_-B_-)^{\frac{1}{2}}}. \tag{23b}$$

In the next section, we shall investigate the criteria to choose the optical solution between Eq.(23a) and Eq.(23b).

III. THE OPTICAL SOLUTIONS

A. The criteria for the optical solutions

In order to calculate the optical solution, we need to determine λ satisfying the cubic equation in Eq.(16). If we introduce parameters

$$\begin{aligned}
q &\equiv (3ac - b^2)/(3a^2), \\
r &\equiv (9abc - 27a^2d - 2b^3)/(27a^3), \\
\Delta &\equiv r^2 + \frac{4}{27}q^3,
\end{aligned} \tag{24}$$

the cubic equation Eq.(16) can be solved easily by the well-known method. That is, if $\Delta > 0$, there are two complex and one real roots, where the real root is expressed as

$$\lambda = -\frac{b}{3a} + \left(\frac{r + \sqrt{\Delta}}{2}\right)^{\frac{1}{3}} + \left(\frac{r - \sqrt{\Delta}}{2}\right)^{\frac{1}{3}}. \tag{25}$$

If $\Delta < 0$, define $\rho \equiv (-\frac{q^3}{27})^{\frac{1}{2}}$ and $\cos \theta \equiv r/(2\rho)$. Then there are three real roots given by

$$\lambda = -\frac{b}{3a} + 2\rho^{\frac{1}{3}} \cos(\frac{\theta}{3}), \quad (26a)$$

$$\lambda = -\frac{b}{3a} + 2\rho^{\frac{1}{3}} \cos(\frac{\theta - 2\pi}{3}), \quad (26b)$$

$$\lambda = -\frac{b}{3a} + 2\rho^{\frac{1}{3}} \cos(\frac{\theta + 2\pi}{3}). \quad (26c)$$

We note that Δ is a sextic polynomial in y as follows.

$$\Delta = D_6 y^6 + D_4 y^4 + D_2 y^2 + D_0, \quad (27)$$

where

$$D_6 \equiv -\frac{256}{27} \epsilon^6 \eta_i^6 (\eta_i - \eta_o)^2 \eta_o^6 m^4 (\eta_o - \eta_i m)^4, \quad (28)$$

$$D_4 \equiv \frac{64 \epsilon^8 \eta_i^4 \eta_o^4 m^4 (\eta_o - \eta_i m)^4 D_4^*}{27(1-m)^2(1+m)^2},$$

$$D_2 \equiv \frac{128 \epsilon^{10} \eta_i^2 \eta_o^2 m^4 (\eta_o - \eta_i m)^4 D_2^*}{27(1-m)(1+m)^4},$$

$$D_0 \equiv -\frac{64}{27} \frac{\epsilon^8 m^4 k^4}{(1+m)^4} [(1+m)\eta_i^2 - 2\eta_i \eta_o]^2 [(1+m)\eta_o^2 - 2m\eta_i \eta_o]^2,$$

and

$$D_4^* = \eta_o^4(8 + 20m^2 - m^4) + 4\eta_i \eta_o^3(-4 - 4m - 15m^2 - 5m^3 + m^4) \quad (29)$$

$$+ 2\eta_i^2 \eta_o^2(2 + 20m + 37m^2 + 20m^3 + 2m^4) - 4\eta_i^3 \eta_o(-1 + 5m + 15m^2 + 4m^3 + 4m^4) \\ + \eta_i^4(-1 + 20m^2 + 8m^4),$$

$$D_2^* = -2\eta_o^6(1+m)^3 + 2\eta_i^6 m^2(1+m)^3 + 2\eta_i \eta_o^5(1+m)^2(2 + 6m + m^2) \quad (30)$$

$$- 2\eta_i^5 \eta_o m(1+m)^2(1 + 6m + 2m^2) + \eta_i^4 \eta_o^2(1 + 11m + 29m^2 + 39m^3 + 18m^4 - 2m^5) \\ + 4\eta_i^3 \eta_o^3(-1 - 2m + m^2 - m^3 + 2m^4 + m^5) \\ - \eta_i^2 \eta_o^4(-2 + 18m + 39m^2 + 29m^3 + 11m^4 + m^5).$$

It follows then that

$$\Delta = -\frac{64}{27} \frac{\epsilon^8 m^4 k^4}{(1+m)^4} \sigma^2 \delta^2 < 0,$$

at $y = 0$ unless $\sigma \cdot \delta = 0$, where σ and δ are defined as

$$\sigma \equiv (1+m)\eta_i^2 - 2\eta_i \eta_o, \quad (31)$$

$$\delta \equiv (1+m)\eta_o^2 - 2m\eta_i \eta_o.$$

Thus unless $\sigma \cdot \delta = 0$, any real root λ in Eq.(26) may be used to calculate the solutions of a Cartesian oval.

We shall show first that the choice of the optical solution between Eq.(23a) and Eq.(23b) is determined by the initial choice of the root λ of Eq.(16).

Now, under the assumption that $\sigma \neq 0$ and $\delta \neq 0$, let us find the roots of Eq.(16) when $y \rightarrow 0$. In fact, from Eq.(18), it follows that $d \rightarrow 0$ as $y \rightarrow 0$. Thus the cubic equation of Eq.(16) in λ becomes $\lambda(a\lambda^2 + b\lambda + c) = 0$. Hence if we expand $\lambda = \lambda_0 + \lambda_1 y + \lambda_2 y^2 + \dots$ when y is small, it is not difficult to find the three real roots λ_0 .

$$\lambda_{01} = 0, \quad (32a)$$

$$\lambda_{02} = 2k^2, \quad (32b)$$

$$\lambda_{03} = \frac{2\epsilon^2 m^2 \eta_i (\eta_i - 2\eta_o + m\eta_i)}{1+m} = \frac{2\epsilon^2 m^2 \sigma}{1+m}. \quad (32c)$$

Also, it is trivial to see that $\lambda_1 = 0$ for each case. Moreover, we may calculate λ_2 corresponding to Eqs.(32a)~(32c) respectively as follows.

$$\lambda_{21} = \frac{2m^2\eta_i\eta_o^2(\eta_o - m\eta_i)^2}{(m-1)(\eta_i - 2\eta_o + m\eta_i)}, \quad (33a)$$

$$\lambda_{22} = \frac{2\eta_i^2\eta_o(\eta_o - m\eta_i)^2}{(m-1)(\eta_o - 2m\eta_i + m\eta_o)}, \quad (33b)$$

$$\lambda_{23} = \frac{-2\eta_i\eta_o(\eta_i - \eta_o)^4m^2(1+m)}{(m-1)(\eta_i - 2\eta_o + m\eta_i)(\eta_o - 2m\eta_i + m\eta_o)}, \quad (33c)$$

where we note that the denominators do not vanish since $\sigma \neq 0$ and $\delta \neq 0$.

Lemma III.1 *Let us assume that $\sigma \neq 0$ and $\delta \neq 0$.*

1. *If (1) $\lambda = \lambda_{01}$ or $\lambda = \lambda_{02}$ at $y = 0$ and $k > 0$, or (2) $\lambda = \lambda_{03}$ at $y = 0$ and $\eta_i > \eta_o$, the optical solution is Eq.(23a).*

2. *If (1) $\lambda = \lambda_{01}$ or $\lambda = \lambda_{02}$ at $y = 0$ and $k < 0$, or (2) $\lambda = \lambda_{03}$ at $y = 0$ and $\eta_i < \eta_o$, the optical solution is Eq.(23b).*

Proof: For small y , we put $B \approx B_0 + B_2y^2$ for $B = B_{\pm}$. Let us consider the case when $\lambda_0 = \lambda_{01}$ or $\lambda_0 = \lambda_{02}$. Then if $k > 0$, a little bit lengthy but straightforward calculation using λ_{21} and λ_{22} in Eq.(33) respectively shows that for small y

$$B_+ \approx \frac{\eta_i - m\eta_o}{\epsilon(1-m)}y^2, \quad (34)$$

while the constant term of B_- in y does not vanish. Thus the optical solution is Eq.(23a) since it passes through the origin. If $k < 0$, however, we have

$$B_- \approx \frac{\eta_i - m\eta_o}{\epsilon(1-m)}y^2, \quad (35)$$

while the constant term of B_+ in y does not vanish. Thus the optical solution is Eq.(23b). On the other hand, let us suppose that $\lambda_0 = \lambda_{03}$. Then if $\eta_i > \eta_o$, we have the same result as that of Eq.(34) for B_+ by a straightforward calculation. Thus the optical solution is Eq.(23a). If $\eta_i < \eta_o$, however, we have the same result as that of Eq.(35) for B_- . Thus the optical solution is Eq.(23b). Thus we have proved the claim. \square

Theorem III.2 *Let us assume that $\sigma \neq 0$ and $\delta \neq 0$.*

Case 1 : Let the root λ be that of Eq.(26a). Then

1-1. *if (1) $\delta > 0$ and $k > 0$ or (2) $\delta < 0$ and $\eta_i > \eta_o$, the optical solution is Eq.(23a),*

1-2. *and if (1) $\delta > 0$ and $k < 0$ or (2) $\delta < 0$ and $\eta_i < \eta_o$, the optical solution is Eq.(23b).*

Case 2 : Let the root λ be that of Eq.(26b). Then

2-1. *if (1) $\delta > 0$, $\sigma > 0$ and $\eta_i > \eta_o$ or (2) $\delta > 0$, $\sigma < 0$ and $k > 0$ or (3) $\delta < 0$ and $k > 0$, the optical solution is Eq.(23a),*

2-2. *and if (1) $\delta > 0$, $\sigma > 0$ and $\eta_i < \eta_o$ or (2) $\delta > 0$, $\sigma < 0$ and $k < 0$ or (3) $\delta < 0$ and $k < 0$, the optical solution is Eq.(23b).*

Case 3 : Let the root λ be that of Eq.(26c). Then

3-1. *if (1) $\delta > 0$, $\sigma > 0$ and $k > 0$ or (2) $\delta > 0$, $\sigma < 0$ and $\eta_i > \eta_o$ or (3) $\delta < 0$ and $k > 0$, the optical solution is Eq.(23a),*

3-2. *and if (1) $\delta > 0$, $\sigma > 0$ and $k < 0$ or (2) $\delta > 0$, $\sigma < 0$ and $\eta_i < \eta_o$ or (3) $\delta < 0$ and $k < 0$, the optical solution is Eq.(23b).*

Proof: Before we prove the claim, let us observe that the arccosine function returns θ with $0 \leq \theta \leq \pi$ in computation. Thus we have

$$\cos\left(\frac{\theta + 2\pi}{3}\right) \leq \cos\left(\frac{\theta - 2\pi}{3}\right) \leq \cos\left(\frac{\theta}{3}\right), \quad (36)$$

where the 1st equality is valid when $\theta = 0$ and the 2nd equality is when $\theta = \pi$.

To prove our claim, it is enough to show which λ_0 in Eq.(32) corresponds to the chosen λ in Eq.(26). Hence Eq.(36) shows the order of the three roots λ in Eq.(26). Now this order can be compared with that of λ_0 in Eq.(32).

It is trivial to see that $\lambda_{01} < \lambda_{02}$ since $k \neq 0$. On the other hand, the difference between λ_{02} and λ_{03} is

$$\begin{aligned}\lambda_{02} - \lambda_{03} &= 2k^2 - \frac{2\epsilon^2 m^2 \eta_i (\eta_i - 2\eta_o + m\eta_i)}{1+m} \\ &= \frac{2\epsilon^2}{1+m} [\eta_o^2 - 2m\eta_i \eta_o + m\eta_o^2] = \frac{2\epsilon^2}{1+m} \delta.\end{aligned}\tag{37}$$

Since $m > 0$, we have

$$\begin{aligned}\lambda_{01} &< \lambda_{03} < \lambda_{02} \text{ if } \delta > 0 \text{ and } \sigma > 0, \\ \lambda_{03} &< \lambda_{01} < \lambda_{02} \text{ if } \delta > 0 \text{ and } \sigma < 0, \\ \lambda_{01} &< \lambda_{02} < \lambda_{03} \text{ if } \delta < 0.\end{aligned}\tag{38}$$

Case 1: Let λ be that of Eq.(26a). Then λ is the largest root of Eq.(16) at $y = 0$. Hence $\lambda = \lambda_{02}$ if $\delta > 0$, and $\lambda = \lambda_{03}$ if $\delta < 0$. Case 2: Let λ be that of Eq.(26b). Then $\lambda = \lambda_{03}$ if $\delta > 0$ and $\sigma > 0$, and $\lambda = \lambda_{01}$ if $\delta > 0$ and $\sigma < 0$, and $\lambda = \lambda_{02}$ if $\delta < 0$. Case 3: Let λ be that of Eq.(26c). Then λ is the smallest root of Eq.(16) at $y = 0$. Thus $\lambda = \lambda_{01}$ if (1) $\delta > 0$ and $\sigma > 0$ or (2) $\delta < 0$, and $\lambda = \lambda_{03}$ if $\delta > 0$ and $\sigma < 0$. Finally, the claim is proved by Lemma 2. \square

Remark: In the case when $\sigma = 0$ or $\delta = 0$, the criteria for the choice of the optical solution between Eq.(23a) and Eq.(23b) is as follows. In fact, if $\sigma = 0$, $\lambda_{01} = \lambda_{03}$, which is a double real root, and λ_{02} is the largest root of Eq.(16). On the other hand, if $\delta = 0$, $\lambda_{02} = \lambda_{03}$, a double real root, and λ_{01} is the smallest root of Eq.(16). For the optical solution to be continuous through $\Delta = 0$, we should choose the single root λ_{02} or λ_{01} for each case since the single root is identical to the root in Eq.(25) in the limits. As seen in Eq.(33), λ_{22} and λ_{21} corresponding to λ_{02} and λ_{01} respectively are still valid only for each $\sigma = 0$ and $\delta = 0$ case. Thus the proof of Lemma 2 is still valid for each case. Hence the optical solution is Eq.(23a) if $k > 0$. It is Eq.(23b) if $k < 0$. In practical computation, however, it would be better to use the root λ in Eq.(25) initially when $\sigma = 0$ or $\delta = 0$.

In the next subsection, we shall discuss more on the continuities of optical solutions.

B. The continuities and interpolations of the optical solutions

Let us suppose that $\delta \neq 0$ and $\sigma \neq 0$. We note that Δ in Eq.(27) is a cubic polynomial in y^2 . Thus we may apply the previous cubic equation solving method to the equation $\Delta = 0$ again. Furthermore, what makes things simple is that not only $\Delta < 0$ when $y = 0$ but also $D_6 < 0$ in Eq.(28). Thus either there is no positive solution y or there is a pair of positive solutions such that $\Delta = 0$.

In fact, from $\tilde{q} \equiv (3D_6 D_2 - D_4^2)/(3D_6^2)$, $\tilde{r} \equiv (9D_6 D_4 D_2 - 27D_6^2 D_0 - 2D_4^3)/(27D_6^3)$ and $\tilde{\Delta} \equiv \tilde{r}^2 + \frac{4}{27}\tilde{q}^3$ corresponding to Eq.(24), it follows that $\tilde{\Delta} > 0$ means that there is no solution y such that $\Delta = 0$. However, if $\tilde{\Delta} < 0$, either there is no positive solution y such that $\Delta = 0$, or there are only two positive solutions y .

If there is no solution y such that $\Delta = 0$, then any initially chosen root λ makes the optical solution continuous for all y on its domain. However, if there is a positive solution y , the initially chosen root λ should be changed as y increases for the continuity of the optical solution. Now we want to choose the root λ such that it is continuous through $\Delta = 0$ when Δ changes its sign. The following claim is very useful in practical computation.

Lemma III.3 *Let $\sigma \neq 0$, $\delta \neq 0$ and y_0 be a positive solution such that $\Delta = 0$.*

1. *If $r > 0$ at y_0 , the only continuous root through $\Delta = 0$ is Eq.(26a).*
2. *If $r < 0$ at y_0 , the only continuous root through $\Delta = 0$ is Eq.(26c).*

Proof: For notational convenience, let us write $\lambda^{(0)}$, $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ respectively for the root λ 's of Eq.(25), Eq.(26a), Eq.(26b) and Eq.(26c). If $r > 0$ at y_0 , $\theta = 0$ or $r = 2\rho$. Now it is easy to see that $\lim_{\Delta \rightarrow 0^-} \lambda^{(1)} = -b/(3a) + 2\rho^{\frac{1}{3}} = \lim_{\Delta \rightarrow 0^+} \lambda^{(0)}$, while $\lim_{\Delta \rightarrow 0^-} \lambda^{(2)} = \lim_{\Delta \rightarrow 0^-} \lambda^{(3)} = -b/(3a) - \rho^{\frac{1}{3}}$. On the other hand, If $r < 0$ at y_0 , $\theta = \pi$ or $r = -2\rho$. Thus $\lim_{\Delta \rightarrow 0^-} \lambda^{(3)} = -b/(3a) - 2\rho^{\frac{1}{3}} = \lim_{\Delta \rightarrow 0^+} \lambda^{(0)}$, while $\lim_{\Delta \rightarrow 0^-} \lambda^{(1)} = \lim_{\Delta \rightarrow 0^-} \lambda^{(2)} = -b/(3a) + \rho^{\frac{1}{3}}$. \square

Remark: In the case when there is a positive solution y such that $\Delta = 0$, we may use the criteria given by Lemma 3 and Theorem 1 together for the initial choices of the root λ and the optical solution of a Cartesian oval. As y increases, the initially chosen root λ should be changed for the continuity of the optical solution. In fact, there is no choice except the root λ of Eq.(25) for the transition from $\Delta < 0$ to $\Delta > 0$. On the other hand, if the transition

direction is reversed, i.e. from $\Delta > 0$ to $\Delta < 0$, there can be three choices in Eq.(26) for the root λ . However, we should choose the root λ according to the above Lemma 3 in order to make the optical solution continuous.

Furthermore, for a continuous optical solution in Eq.(23a) or Eq.(23b)

$$z = B/[1 + (1 - AB)^{\frac{1}{2}}] \quad (A \equiv A_{\pm} \text{ and } B \equiv B_{\pm}), \quad (39)$$

it might happen that $1 - AB < 0$ as y increases. In this case, we can interpolate it continuously by putting $1 - AB = 0$ in a manner similar to conic case in optical design. That is, we may interpolate the optical solution in Eq.(39) by the curve $z = B$ in the region where $1 - AB < 0$.

Definition III.4 *Let $z = B/[1 + (1 - AB)^{\frac{1}{2}}]$ be an optical solution. Then $z = B$ is called an interpolating curve of the optical solution in the region where $1 - AB < 0$.*

We conclude this section with the observation that the coefficient of the second order term in y of the optical solution is the curvature c_0 of the optical solution from Eqs.(34, 35)

$$c_0 \equiv \frac{\eta_i - m\eta_o}{\epsilon(1 - m)}. \quad (40)$$

When y is small, it is also interesting to see that the optical solution is of the following form.

$$z = \frac{c_0 y^2 + O(y^4)}{1 + \sqrt{1 - (1 + K)c_0^2 y^2 + O(y^4)}} \quad (41)$$

for some constant K as a function of constants η_i, η_o and m . This form for the optical solution of a Cartesian oval shows its deviation from a conic curve when y is small. It looks like that K plays the role of a conic constant. However, the insightful relations of the optical solutions to conics shall be shown in the next section.

IV. SUPERCONICS AS EXTENSIONS OF ASPHERIC CURVES BASED ON CONICS

It is well-known in the literature[2, 3, 7] that Cartesian ovals become conics if $m = \pm 1$, which can be easily observed in Eq.(9) of this work. In this section, however, we shall consider the limits of the optical solution expressed in Eq.(23a) or Eq.(23b) from a different point of view.

In fact, we want to see the limits of both optical solutions and their interpolating curves. First of all, we observe that Eq.(9) is invariant under the replacements of η_i, η_o and m by η_o, η_i and $1/m$ respectively:

$$\eta_i \Rightarrow \eta_o, \quad \eta_o \Rightarrow \eta_i, \quad m \Rightarrow 1/m. \quad (42)$$

Thus any optical solution with $0 < m < 1$ may be represented by an optical solution with $m > 1$. From now on, we may assume that $m > 1$ without loss of generality.

Theorem IV.1 *Let $m > 1$ and $z = B/[1 + (1 - AB)^{\frac{1}{2}}]$ be an optical solution. Then we have*

$$\begin{aligned} \lim_{\eta_i \rightarrow 0} \frac{B}{1 + (1 - AB)^{\frac{1}{2}}} &= \frac{c_0 y^2}{1 + [1 - (1 + K)c_0^2 y^2]^{\frac{1}{2}}} && \text{if } 1 - AB \geq 0, \\ \lim_{\eta_i \rightarrow 0} B &= c_0 y^2 && \text{if } 1 - AB < 0, \end{aligned} \quad (43)$$

where the curvature c_0 and the conic constant K are defined as

$$c_0 \equiv \frac{\eta_o}{\epsilon(1 - \frac{1}{m})}, \quad K \equiv -\frac{1}{m^2}. \quad (44)$$

Proof: Let us choose the root λ of Eq.(26a) to make the optical solution continuous for all y as can be seen in the below. Now for small η_i we put

$$\rho^{\frac{1}{3}} = \sqrt{-\frac{q}{3}} \equiv [\rho^{\frac{1}{3}}, 0] + [\rho^{\frac{1}{3}}, 1]\eta_i + [\rho^{\frac{1}{3}}, 2]\eta_i^2,$$

where the notation $[f, n]$ represents the n th order coefficient of f in η_i . Then it is not hard to find

$$\begin{aligned} [\rho^{\frac{1}{3}}, 0] &= \frac{2}{3}\epsilon^2\eta_o^2, \\ [\rho^{\frac{1}{3}}, 1] &= -\frac{2(2+m)m\epsilon^2\eta_o}{3(1+m)}, \\ [\rho^{\frac{1}{3}}, 2] &= \frac{\epsilon^2m^2(1+2m+4m^2)}{3(1+m)^2} - \frac{2}{3}(1+2m^2)\eta_o^2y^2. \end{aligned} \quad (45)$$

Now we put

$$\cos \frac{\theta}{3} \equiv [\cos \frac{\theta}{3}, 0] + [\cos \frac{\theta}{3}, 1]\eta_i + [\cos \frac{\theta}{3}, 2]\eta_i^2. \quad (46)$$

From the observations that $\cos \theta = 4 \cos^3 \frac{\theta}{3} - 3 \cos \frac{\theta}{3}$ and

$$\begin{aligned} [\cos \theta, 0] &= 1, \\ [\cos \theta, 1] &= 0, \\ [\cos \theta, 2] &= -\frac{27\epsilon^4\eta_o^4m^4[\epsilon^2(m-1) - (1+m)\eta_o^2y^2]}{2(m-1)(1+m)^2\epsilon^6\eta_o^6}, \end{aligned} \quad (47)$$

we have

$$\begin{aligned} [\cos \frac{\theta}{3}, 0] &= 1, \\ [\cos \frac{\theta}{3}, 1] &= 0, \\ [\cos \frac{\theta}{3}, 2] &= -\frac{3m^4}{2(1+m)^2\eta_o^2} + \frac{3m^4y^2}{2\epsilon^2(m-1)(1+m)}. \end{aligned} \quad (48)$$

Thus if we put

$$\lambda \equiv [\lambda, 0] + [\lambda, 1]\eta_i + [\lambda, 2]\eta_i^2,$$

it is straightforward from Eq.(26a) and Eqs.(45, 48) to see the followings

$$\begin{aligned} [\lambda, 0] &= 2\epsilon^2\eta_o^2 \\ [\lambda, 1] &= -4m\epsilon^2\eta_o. \\ [\lambda, 2] &= 2\epsilon^2m^2 + \frac{2}{m^2-1}\eta_o^2y^2. \end{aligned} \quad (49)$$

For $y = 0$, $\lambda \approx 2\epsilon^2\eta_o^2 - 4m\epsilon^2\eta_o\eta_i + 2\epsilon^2m^2\eta_i^2 = 2k^2$ which corresponds to λ_{02} in Eq.(32b). Now we put $A = A_0 + A_1\eta_i + \dots$, and $B = B_0 + B_1\eta_i + \dots$ and suppose that $\eta_o > 0$. Then since $\delta \approx (1+m)\eta_o^2 > 0$ and $k = \epsilon(\eta_o - m\eta_i) = \epsilon\eta_o > 0$, the use of $A = A_+$ and $B = B_+$ according to the criteria in Theorem 1 yields

$$A_0 = \frac{(1+m)\eta_o}{\epsilon m} = (1+K)c_0, \quad B_0 = \frac{m\eta_o y^2}{\epsilon(m-1)} = c_0 y^2. \quad (50)$$

On the other hand, suppose that $\eta_o < 0$. Then $k < 0$ and we obtain the same result for $A = A_-$ and $B = B_-$. Hence the case when $1 - AB \geq 0$ has been proved.

Now we observe from Eq.(27) that, when $\eta_i \rightarrow 0$,

$$\Delta \approx -\frac{256\epsilon^{10}m^4\eta_o^{10}}{27(1+m)^2(m-1)}((m-1)\epsilon^2 - (1+m)\eta_o^2y^2)\eta_i^2. \quad (51)$$

Thus the root of Eq.(26) for $\Delta < 0$ is changed once to that of Eq.(25) at $y = \sqrt{\frac{(m-1)\epsilon^2}{(m+1)\eta_o^2}} = \sqrt{\frac{1}{(1+K)c_0^2}}$ through $\Delta = 0$. Hence it remains to show the limit when $\Delta > 0$.

When $\eta_i \rightarrow 0$, let us put $\lambda \approx \lambda_0 + \lambda_1 \eta_i + \lambda_2 \eta_i^2$ again for $\Delta > 0$. Then from Eq.(25), the same coefficients as in Eq.(49) are obtained by a little bit lengthy calculation. Hence we have $B \rightarrow c_0 y^2$ for $y \geq \sqrt{\frac{1}{(1+K)c_0^2}}$ or $1 - AB = 1 - (1+K)c_0^2 y^2 \leq 0$. \square

Any central conic with $c_0, K (0 < K < 1)$ determines η_o, m by Eq.(44). We remark that the circle is obtained in the limit when $m \rightarrow \infty$ and the parabola is the limit of the central conic when $m \rightarrow 1$ with c_0 being fixed.

Theorem IV.2 *Let $m > 1$ and $z = B/[1 + (1 - AB)^{\frac{1}{2}}]$ be an optical solution. Then we have*

$$\lim_{\eta_o \rightarrow 0} \frac{B}{1 + (1 - AB)^{\frac{1}{2}}} = \frac{c_0 y^2}{1 + [1 - (1 + K)c_0^2 y^2]^{\frac{1}{2}}} \text{ for all } y, \quad (52)$$

where the curvature c_0 and the conic constant K are defined as

$$c_0 \equiv \frac{\eta_i}{\epsilon(1 - m)}, \quad K \equiv -m^2. \quad (53)$$

Proof: On the contrary to the case when $\eta_i \rightarrow 0$,

$$\Delta \approx -\frac{256\epsilon^{10}m^{10}\eta_i^{10}}{27(1+m)^2(m-1)}((m-1)\epsilon^2 + (1+m)\eta_i^2 y^2)\eta_o^2, \quad (54)$$

when $\eta_o \rightarrow 0$. Thus $\Delta < 0$ for all y since $m > 1$. That is, there is no solution y such that $\Delta = 0$. Hence we may choose the root of Eq.(26a) for the convenience of computation.

In a similar manner as in the previous theorem, we put

$$\rho^{\frac{1}{3}} = \sqrt{-\frac{q}{3}} \equiv [\rho^{\frac{1}{3}}, 0] + [\rho^{\frac{1}{3}}, 1]\eta_o + [\rho^{\frac{1}{3}}, 2]\eta_o^2,$$

where the notation $[f, n]$ represents the n th order coefficient of f in η_o . Then it is not hard to find

$$\begin{aligned} [\rho^{\frac{1}{3}}, 0] &= \frac{2}{3}\epsilon^2 m^2 \eta_i^2, \\ [\rho^{\frac{1}{3}}, 1] &= -\frac{2(1+2m)m\epsilon^2 \eta_i}{3(1+m)}, \\ [\rho^{\frac{1}{3}}, 2] &= \frac{\epsilon^2(4+2m+m^2)}{3(1+m)^2} - \frac{2}{3}(2+m^2)\eta_i^2 y^2. \end{aligned} \quad (55)$$

Now we observe that

$$\begin{aligned} [\cos \theta, 0] &= -1, \\ [\cos \theta, 1] &= 0, \\ [\cos \theta, 2] &= Q, \end{aligned} \quad (56)$$

where for $m > 1$

$$Q \equiv \frac{27}{2m^2(1+m)^2\eta_i^2} + \frac{27y^2}{2\epsilon^2 m^2(m^2 - 1)} > 0. \quad (57)$$

We put

$$\cos \frac{\theta}{3} \equiv [\cos \frac{\theta}{3}, 0] + [\cos \frac{\theta}{3}, 1]\eta_o + [\cos \frac{\theta}{3}, 2]\eta_o^2. \quad (58)$$

Then from the observations that $\cos \theta = 4 \cos^3 \frac{\theta}{3} - 3 \cos \frac{\theta}{3}$ and $\cos(\arccos(-1 + cx^2)/3) = \frac{1}{2} + \sqrt{\frac{c}{6}}x - \frac{c}{18}x^2 + \dots$ for $c \geq 0$ and some x , it follows that

$$\begin{aligned} [\cos \frac{\theta}{3}, 0] &= \frac{1}{2}, \\ [\cos \frac{\theta}{3}, 1] &= \sqrt{\frac{Q}{6}}, \\ [\cos \frac{\theta}{3}, 2] &= -\frac{Q}{18}. \end{aligned} \quad (59)$$

Thus if we put

$$\lambda \equiv [\lambda, 0] + [\lambda, 1]\eta_o + [\lambda, 2]\eta_o^2,$$

it is straightforward from Eq.(26a) and Eqs.(55, 59) to see the followings

$$\begin{aligned} [\lambda, 0] &= 2\epsilon^2 m^2 \eta_i^2, \quad [\lambda, 1] = -\frac{2(1+2m)m\epsilon^2 \eta_i}{1+m} + \frac{4\epsilon^2 m^2 \eta_i^2}{3\sqrt{6}} \sqrt{Q}, \\ [\lambda, 2] &= -\frac{2(m^2-1)(\epsilon^2 + (m^2-1)\eta_i^2 y^2)}{3(1-m^2)} + \epsilon^2 m^2 \eta_i^2 \cdot \left[\frac{1}{m^2(1+m)^2 \eta_i^2} + \frac{y^2}{\epsilon^2 m^2 (m^2-1)} \right] \\ &\quad + \left[\frac{\epsilon^2(4+2m+m^2)}{3(1+m)^2} - \frac{2}{3}(2+m^2)\eta_i^2 y^2 \right] - \frac{2(1+2m)m\epsilon^2 \eta_i}{3(1+m)} \cdot \frac{1}{\sqrt{6}} \sqrt{Q}. \end{aligned} \quad (60)$$

The optical solution is Eq.(23a) as can be seen in the below. We expand A_+ and B_+ in terms of η_o : $A = A_0 + A_1\eta_o + \dots$, and $B = B_0 + B_1\eta_o + \dots$. Then it is lengthy but straightforward to obtain the following.

$$A_0 = 0, \quad A_1 = \frac{m \pm 1}{\epsilon m}, \quad B_0 = \frac{2\epsilon}{(1+m)\eta_i} \pm \frac{2\sqrt{6}\epsilon m \sqrt{Q}}{9}, \quad (61)$$

where $(+)$ and $(-)$ are for $\eta_i < 0$ and $\eta_i > 0$ respectively. Thus we have

$$\begin{aligned} \lim_{\eta_o \rightarrow 0} \frac{B}{1 + (1-AB)^{\frac{1}{2}}} &= \frac{B_0}{2} = \frac{\epsilon}{(1+m)\eta_i} \pm \frac{\sqrt{6}\epsilon m \sqrt{Q}}{9} \\ &= \frac{1}{(1+K)c_0} \pm \frac{\epsilon}{(1+m)|\eta_i|} \sqrt{1 - (1+K)c_0^2 y^2} \\ &= \frac{c_0 y^2}{1 + [1 - (1+K)c_0^2 y^2]^{\frac{1}{2}}}. \end{aligned} \quad (62)$$

Thus the claim has been proved. \square

Any central conic with $c_0, K (K > 1)$ determines η_i, m by Eq.(53). In this case, the parabola is the limit of the central conic when $m \rightarrow 1$ with c_0 being fixed.

Definition IV.3 A superconic curve is defined to be an aspheric curve based on optical solution described as follows: For any y ,

$$z = B/[1 + (1-AB)^{\frac{1}{2}}] + \sum_{n=2}^N f_{2n} y^{2n}, \quad (63)$$

where N is a positive integer and f_{2n} 's are some constants.

The superconic curve is obviously different from Greynolds'[11]. Now we have the main result of this work, which follows obviously from the previous theorems.

Corollary IV.4 A superconic curve described by Eq.(63) is an extension of an aspheric curve based on conic described by Eq.(3).

V. EXAMPLE : A FAMILY OF OPTICAL SOLUTIONS AND CONICS

We note that $m(m > 1)$ and η_i, η_o are free variables for the optical solutions of Cartesian ovals. However, there may be some other choices for free variables. It seems to be nice to choose the curvature c_0 , m and η_i as free variables, in which case η_o can be obtained by $\eta_o = (\eta_i - \epsilon(1-m)c_0)/m$ from Eq.(40).

With this choice of free variables, we may describe the set of curves with the same curvature c_0 and m but different η_i 's as one family. Of course, typical members of the family are the optical solutions with such c_0 and m . However, it is interesting that there are some special members in the family. That is, if $\eta_i = 0$ or $\eta_i = \epsilon(1-m)c_0$ (i.e. $\eta_o = 0$), the curves become conics. In fact, when $\eta_i = 0$, the curve is an ellipse with the curvature c_0 and the conic constant $K = -1/m^2$. When $\eta_i = \epsilon(1-m)c_0$, the curve is a hyperbola with the same curvature c_0 and the conic constant $K = -m^2$.

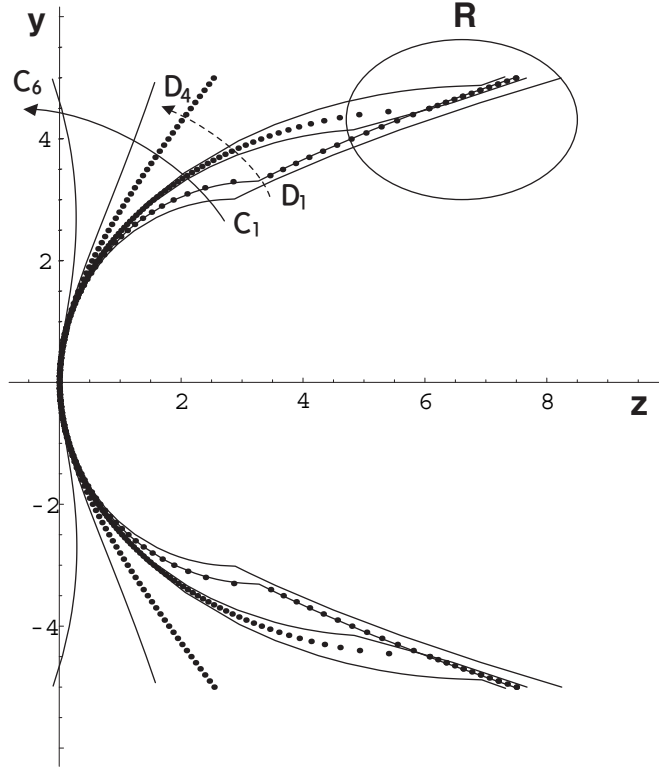


FIG. 1: The Figure shows a family of curves containing of optical solutions C_1, \dots, C_6 (real lines) and conics $D_1(=D_2)$, D_3 and D_4 (dotted lines) following the arrows respectively with the same curvature $c_0 = 0.3$ and $m = 1.5$, but with different η_i 's. The values of η_i 's of the curves are $\eta_i = 0.3(C_1)$, $0.18(D_1)$, $0.15(C_2)$, $0.072(D_2)$, $0.01(C_3)$, $0(D_3)$, $-0.01(C_4)$, $-0.09(D_4)$, $-0.15(C_5)$ and $-0.3(C_6)$ respectively. The interpolating curves of the conics D_1, D_2 and D_3 are $z = c_0 y^2$ when $1 - (1 + K)c_0^2 y^2 < 0$. Similarly, the interpolating curves of the optical solutions C_1, \dots, C_4 are $z = B$ when $1 - AB < 0$. They are drawn in the region R .

Especially, as can be seen from Eq.(9), if $\eta_i = \epsilon c_0$ (i.e. $\eta_i = \eta_o$) or $\eta_i = \epsilon c_0 / (1 + m)$ (i.e. $\eta_i = \eta_o / m$ or $k = 0$), they represent one circle with the same curvature c_0 and the conic constant $K = 0$. It may be easily observed that it is the circle possessed commonly by all the families with the same curvature c_0 . Thus even conic curves with such special η_i 's in the above may be regarded as members of the family of curves with c_o and m .

In Fig. 1, we demonstrate an example of a family of curves with the same curvature $c_0 = 0.3$ and $m = 1.5$ for $\epsilon = 0.6$. The family contains the optical solutions with the same c_0 and m but with different η_i 's in the range $0.3(C_1) \sim -0.3(C_6)$ following the arrow.

Moreover, the family contains the conics as well with the same curvature c_0 and m but with different η_i 's in the range $0.18(D_1) \sim -0.09(D_4)$. Here we note that the conics D_1 and D_2 are the same circle with $\eta_i = \epsilon c_0 = 0.18$ and $\eta_i = \epsilon c_0 / (1 + m) = 0.072$ respectively, which is a peculiar fact. It is observed that the optical solutions, e.g. C_2 , with η_i 's between 0.18 and 0.072 are very close to the circle.

The curve D_3 is an ellipse with $\eta_i = 0$. The conic constant is $K = -1/m^2 \approx -0.44$. The optical solutions with small η_i such as C_3 or C_4 are slightly deviated from the conic curve D_3 . If η_i is not small, e.g. C_1, C_6 , it is far different from the conic curve although it has the same c_0 and m .

The curve D_4 is the case when $\eta_i = \epsilon(1 - m)c_0 = -0.09$. It is interesting to note that D_4 is a hyperbola with the conic constant $K = -m^2 = -2.25$ by Eq.(53) since $\eta_o = 0$. In fact, there always exists such a pair of conics as an ellipse D_3 and a hyperbola D_4 for the family with a given c_0 and m , and both of them approach one parabola when $K \rightarrow -1$ (i.e. $m \rightarrow 1$).

The interpolating curves of the optical solutions are described by $z = B$ like the interpolating conic curves are described by $z = c_0 y^2$ for the case when $1 - (1 + K)c_0^2 y^2 < 0$ in Eq.(2). The interpolating curves are shown in the region R .

For most of the optical solutions in Fig.1, the initially chosen root λ of Eq.(26a) has been changed to that of Eq.(25) at some y . Furthermore, for the curves C_1 and C_2 , the root λ has been changed again even to that of Eq.(26c) in the region R .

It is natural to extend the family of optical solutions and conics to the family of superconic curves and aspheric curves based on conics by adding the higher order polynomial terms $\sum_{n=2}^N f_{2n}y^{2n}$ to optical solutions and conics as in Eq.(63) and Eq.(3).

The optical solutions may be used as the starting curves for lenses at the initial design of an optical system using the perfectly focusing property. And the initial optical design may be followed by an elaborate optimization, in which case superconic curves in Eq.(63) seem to be the most suitable curves for the optimization since they are extensions of not only the initial optical solutions but also aspheric curves based on conics in Eq.(3).

VI. CONCLUSIONS

In this work, we have investigated the criteria to find the optical solution of a Cartesian oval and discussed the continuity and interpolation of the optical solution. Moreover, we have shown that conics and their interpolating curves are the limiting cases of the optical solutions and their interpolating curves respectively. It follows then that all those curves with the same curvature c_0 and parameter m but different η_i 's, including both optical solutions and conics, can be regarded as members of one family of curves. We have demonstrated an example about a family of curves.

Most of all, the above work on the optical solutions makes it possible to construct another kind of superconic curves in Eq.(63) that are different from Greynolds'. That is, the superconic curves suggested in this work are extensions of not only conics and Cartesian ovals but also aspheric curves based on conics in Eq.(3) while Greynolds' superconic curves are not extensions of aspheric curves based on conics. Also the superconic curves in this work are expressed in explicit form as in Eq.(63) while Greynolds' superconic curves are in implicit form as in Eq.(4).

The relationship between superconic curves in Eq.(63) and aspheric curves based on conics in Eq.(3) seems to be a promising property that Greynolds' superconic curves do not have.

Acknowledgments

The author would like to thank Professor B. S. Lee for helpful discussions and critical comments. This paper was supported by the Semyung University Research Grant of 2013.

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